Tutorial Notes 12

1. Find the outward flux of $F = (x^2, y^2, z^2)$ across the boundaries of the following regions:

   (a) $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$;

   (b) $-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1$;

   (c) $x^2 + y^2 \leq 4, 0 \leq z \leq 1$.

   Solutions:

   $\operatorname{div} F = 2x + 2y + 2z$. By the divergence theorem, it suffices calculate the integrals of $\operatorname{div} F$ on these regions, which are denoted by Ω_1, Ω_2, Ω_3.

   (a)
   $$\int_{\Omega_1} (2x + 2y + 2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 6\int_{\Omega_1} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 6\int_0^1\int_0^1\int_0^1 x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 3.$$

   (b) By symmetry, the integral vanishes.

   (c) By symmetry,
   $$\int_{\Omega_3} x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{\Omega_3} y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0.$$
   It suffices to calculate
   $$\int_{\Omega_3} 2z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$
   which is equal to, in cylindrical coordinates,
   $$\int_0^{2\pi}\int_0^2\int_0^1 2zr\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = 4\pi.$$

2. Find the outward flux of $F = (2xz, -xy, -z^2)$ across the boundary of the region: $y + z \leq 4$, $4x^2 + y^2 \leq 16$, $x, y, z \geq 0$.

   Solutions:

   $\operatorname{div} F = -x$. By the divergence theorem, it suffices to calculate
   $$\int_0^4\int_0^{4-y}\int_0^{\sqrt{4-y^2/4}} -x\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y,$$

which is equal to

$$\int_0^4 \int_0^{4-y} \left(\frac{y^2}{8} - 2\right) \mathrm{d}z \, \mathrm{d}y = \int_0^4 (4-y) \left(\frac{y^2}{8} - 2\right) \mathrm{d}y = -\frac{40}{3}.$$

3. (a) Prove that the outward flux of $F = (x, y, z)$ across a boundary of a region is three times the volume of the region.

   (b) Prove that for a smooth closed surface, F can not be orthogonal to the normal vector everywhere.

   Solutions:

   (a) Due to $\operatorname{div} F = 3$ and the divergence theorem, the conclusion follows immediately.

   (b) Assume that F is orthogonal to the normal vector everywhere, then the outward flux of F vanishes, which is a contradiction.

4. Consider the curved cube Q: $0 \le z \le f(x, y)$, $0 \le x \le 1$, $0 \le y \le 1$. Suppose that $F = (x, -2y, z + 3)$ and the outward flux of F across the side $x = 1$ is 1 and across the side $y = 1$ is -3. Find the outward flux across the top.

   Solutions:

   Since $\operatorname{div} F = 0$, according to the divergence theorem, the total outward flux of F is 0. Hence it suffices to calculate the outward fluxes across the other five sides. We have known the fluxes across the sides $x = 1$, $y = 1$, so we only need to find the fluxes across the sides $x = 0$, $y = 0$, $z = 0$, which are denoted by S_1, S_2, S_3. We have

$$\int_{S_1} F \cdot n \, \mathrm{d}S = 0;$$
$$\int_{S_2} F \cdot n \, \mathrm{d}S = 0;$$
$$\int_{S_3} F \cdot n \, \mathrm{d}S = -3.$$

   Therefore the flux of the top is 5.

5. Prove the vector field identities:

   (a) $\nabla \times (F_1 \times F_2) = (F_2 \cdot \nabla)F_1 + (\nabla \cdot F_2)F_1 - (F_1 \cdot \nabla)F_2 - (\nabla \cdot F_1)F_2$;

   (b) $\nabla(F_1 \cdot F_2) = (F_1 \cdot \nabla)F_2 + (F_2 \cdot \nabla)F_1 + F_1 \times (\nabla \times F_2) + F_2 \times (\nabla \times F_1)$,

   where $(F \cdot \nabla)G$ means directional derivatives along F for each component of G.

   Solutions:

   To make the proof clear we use X and Y to denote the two vector fields and to simplify the notations we use $f_{,i}$ to denote $\partial_i f$. Moreover, since the three components are similar, we only examine the first component.

(a) First

$$X \times Y = (X^2Y^3 - X^3Y^2, X^3Y^1 - X^1Y^3, X^1Y^2 - X^2Y^1).$$

Then it follows that

$$\begin{aligned}
&[\nabla \times (X \times Y)]_1 \\
&= (X^1Y^2 - X^2Y^1)_{,2} - (X^3Y^1 - X^1Y^3)_{,3} \\
&= Y^2X^1_{,2} + Y^2_{,2}X^1 - X^2Y^1_{,2} - X^2_{,2}Y^1 - X^3Y^1_{,3} - X^3_{,3}Y^1 + Y^3X^1_{,3} + Y^3_{,3}X^1 \\
&= (Y^2X^1_{,2} + Y^3X^1_{,3}) + (Y^2_{,2}X^1 + Y^3_{,3}X^1) \\
&\quad - (X^2Y^1_{,2} + X^3Y^1_{,3}) - (X^2_{,2}Y^1 + X^3_{,3}Y^1) \\
&= (Y \cdot \nabla)X^1 + (\nabla \cdot Y)X^1 - Y^1X^1_{,1} - Y^1_{,1}X^1 \\
&\quad - (X \cdot \nabla)Y^1 - (\nabla \cdot X)Y^1 + X^1Y^1_{,1} + X^1_{,1}Y^1 \\
&= (Y \cdot \nabla)X^1 + (\nabla \cdot Y)X^1 - (X \cdot \nabla)Y^1 - (\nabla \cdot X)Y^1.
\end{aligned}$$

(b) First

$$\nabla \times Y = (Y^3_{,2} - Y^2_{,3}, Y^1_{,3} - Y^3_{,1}, Y^2_{,1} - Y^1_{,2}).$$

Then

$$\begin{aligned}
[X \times (\nabla \times Y)]_1 &= X^2(Y^2_{,1} - Y^1_{,2}) - X^3(Y^1_{,3} - Y^3_{,1}) \\
&= X^2Y^2_{,1} + X^3Y^3_{,1} - (X \cdot \nabla)Y^1 + X^1Y^1_{,1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[Y \times (\nabla \times X)]_1 &= Y^2(X^2_{,1} - X^1_{,2}) - Y^3(X^1_{,3} - X^3_{,1}) \\
&= Y^2X^2_{,1} + Y^3X^3_{,1} - (Y \cdot \nabla)X^1 + Y^1X^1_{,1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&[X \times (\nabla \times Y)]_1 + [Y \times (\nabla \times X)]_1 + (X \cdot \nabla)Y^1 + (Y \cdot \nabla)X^1 \\
&= X^1Y^1_{,1} + X^2Y^2_{,1} + X^3Y^3_{,1} + Y^1X^1_{,1} + Y^2X^2_{,1} + Y^3X^3_{,1} \\
&= (X \cdot Y)_{,1}.
\end{aligned}$$

Remark 1

Another method could also deal with vector field identities. First we have the following observations. Let ε_{ijk} be the sign of the permutation (ijk) (if (ijk) contains repeated elements, then $\varepsilon_{ijk} = 0$). Then

$$(X \times Y)_i = \sum_{jk} \varepsilon_{ijk} X^j Y^k,$$

$$(\nabla \times X)_i = \sum_{jk} \varepsilon_{ijk} X^k_{,j}.$$

Moreover

$$\sum_i \varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks},$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Now we use these facts to prove (a) as an example.

$$\begin{aligned}
[\nabla \times (F^1 \times F^2)]_i &= \sum_{jk} \varepsilon_{ijk} (F^1 \times F^2)^k_{,j} \\
&= \sum_{jkst} \varepsilon_{ijk}\varepsilon_{kst} (F^1_s F^2_t)_{,j} \\
&= \sum_{jst} (\delta_{is}\delta_{jt} - \delta_{it}\delta_{js})(F^1_s F^2_t)_{,j} \\
&= \sum_j (F^1_i F^2_j)_{,j} - \sum_j (F^1_j F^2_i)_{,j} \\
&= \sum_j F^2_{j,j} F^1 + \sum_j F^2_j F^1_{i,j} - \sum_j F^1_{j,j} F^2_i - \sum_j F^1_j F^2_{i,j} \\
&= (\nabla \cdot F^2) F^1_i + (F^2 \cdot \nabla) F^1_i - (\nabla \cdot F^1) F^2_i - (F^1 \cdot \nabla) F^2_i.
\end{aligned}$$